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A sufficient condition for graphs to be fractional (k, m)-deleted graphs☆

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ABSTRACT

Let G be a graph, and k a positive integer. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is called a fractional (k, m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. In this paper, we use a binding number to obtain a sufficient condition for a graph to be a fractional (k, m)-deleted graph. This result is best possible in some sense, and it is an extension of Zhou's previous results.

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1. Introduction

Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorisations in networks are very useful in combinatorial design, network design, circuit layout, and so on. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term "graph" instead of "network".

We investigate the fractional factor problem in graphs, which can be considered as a relaxations of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

All graphs considered are finite undirected graphs which have neither multiple edges nor loops. We refer the readers to [1] for the terminology not defined here. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we use $N_G(x)$ for the set of vertices of $V(G)$ adjacent to x, and $d_G(x)$ for the degree of x in G. Set $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. For any $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{x \in S} N_G(x)$. Note that $N_G(x)$ does not contain x, but it may happen that $N_G(S) \supseteq S$. The subgraphs of G induced by S and $V(G) \setminus S$ are denoted by $G[S]$ and $G - S$, respectively. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be two disjoint subsets of $V(G)$, we use $E_G(S, T)$ to denote the set of edges with one end in S and the other end in T. Write $e_G(S, T) = |E_G(S, T)|$. The binding number of G is defined as

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

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Let $k \geq 1$ be an integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching [2]. Zhou [3] introduced first the definition of a fractional (k, m)-deleted graph, that is, a graph G is called a fractional (k, m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. A fractional (k, m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

Katerinis and Woodall [4] gave a binding number condition for a graph to have a k-factor. Zhou [5,6] obtained some sufficient conditions for graphs to have factors. Correa and Matamala [7] showed a new necessary and sufficient condition for graphs to have factors. Liu and Zhang [8] showed a toughness condition for graphs to have fractional k-factors. Zhou [9–12] obtained some sufficient conditions for graphs to have fractional k-factors. Yu [13] gave a degree condition for graphs to have fractional k-factors. Liu and Zhang [14] investigated fractional k-factors of graphs. Zhou [3,15] obtained two sufficient conditions for graphs to be fractional (k, m)-deleted graphs.

The following results on k-factors, fractional k-factors and fractional (k, m)-deleted graphs are known.

Theorem 1 (*Katerinis and Woodall [4]*)**.** *Let k be an integer such that $k \geq 2$, and let G be a graph of order n such that $n \geq 4k - 6$, kn is even, and $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$. Then G has a k-factor.*

Theorem 2 (*Liu and Zhang [8]*)**.** *Let $k \geq 2$ be an integer. A graph G of order n with $n \geq k + 1$ has a fractional k-factor if its toughness $t(G) \geq k - \frac{1}{k}$.*

Theorem 3 (*Zhou and Duan [11]*)**.** *Let $k \geq 2$ be an integer, and let G be a graph of order n such that $n \geq 4k - 6$. Then*

(1) *If kn is even, and $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$, then G has a fractional k-factor; and*

(2) *If kn is odd, and $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)+2}$, then G has a fractional k-factor.*

Theorem 4 (*Zhou [3]*)**.** *Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2-1}{4k}$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

Theorem 5 (*Zhou [15]*)**.** *Let $k \geq 1$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 4k - 3 + 2(2k + 1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2-3}{4k}$. If*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

In this paper, we use binding number to obtain a new sufficient condition for a graph to be a fractional (k, m)-deleted graph. Our result is the following theorem which is an extension of Theorem 3.

Theorem 6. *Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 4k - 6 + \frac{2m}{k-1}$. If*

$$\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2) - 2m + 2},$$

then G is a fractional (k, m)-deleted graph.

If $m = 0$ in Theorem 6, then we obtain the following corollary.

Corollary 1. *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k - 6$. If*

$$\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2) + 2},$$

then G has a fractional k-factor.

If $m = 1$ in Theorem 6, then we get the following corollary.

Corollary 2 (*Zhou [16]*)**.** *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k - 5$. If*

$$\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)},$$

then G is a fractional k-deleted graph.

2. The proof of Theorem 6

In order to prove Theorem 6, we depend on the following lemmas.

Lemma 2.1 (*Woodall [17]*). *Let G be a graph of order n with* $\operatorname{bind}(G) > c$. *Then* $\delta(G) > n - \frac{n-1}{c}$.

Lemma 2.2 (*Zhou [3]*). *Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k, m)-deleted graph if and only if*

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T)$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 6. Suppose that G satisfies the conditions of Theorem 6, but is not a fractional (k, m)-deleted graph. According to Lemma 2.2 there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \leq \sum_{x \in T} d_H(x) - e_H(S, T) - 1, \tag{1}$$

where H is some subgraph of G with m edges. Since $|E(H)| = m$, we have $\sum_{x \in T} d_H(x) - e_H(S, T) \leq 2m$. Thus, according to (1) we obtain

$$\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \leq 2m - 1. \tag{2}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$ by (1).

Claim 1. $d_{G-S}(x) \leq k - 1$ *for any* $x \in T$.

Proof. If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2). This contradicts the choice of S and T. The proof of Claim 1 is complete. □

Set

$$h = \min\{d_{G-S}(x) : x \in T\},$$

and choose $x_1 \in T$ with $d_{G-S}(x_1) = h$. In terms of Claim 1, we have $0 \leq h \leq k - 1$. Obviously, the following inequalities hold.

$$\delta(G) \leq d_G(x_1) \leq d_{G-S}(x_1) + |S| = h + |S|,$$

that is,

$$|S| \geq \delta(G) - h. \tag{3}$$

Using (3), Lemma 2.1 and $\operatorname{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)-2m+2}$, we have

$$|S| > \frac{(k-1)(n+2) + 2m}{2k-1} - h. \tag{4}$$

Now in order to prove the theorem, we shall deduce some contradictions by the following three cases.
Case 1. $2 \leq h \leq k - 1$.
Subcase 1.1. $|T| < \frac{k(n-2)-2m+2}{2k-1} + h$.
In this case, it is easy to see that

$$|T| \leq \frac{k(n-2) - 2m + 1}{2k-1} + h. \tag{5}$$

From (4), we obtain

$$|S| \geq \frac{(k-1)(n+2) + 2m + 1}{2k-1} - h. \tag{6}$$

In terms of (2), we have

$$-1 \geq \delta_G(S, T) - 2m = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| - 2m \geq k|S| - (k-h)|T| - 2m. \tag{7}$$

Multiplying (7) by $(2k-1)$ and rearranging, and then using (5) and (6),

$$\begin{aligned}0 &\geq k(2k-1)|S| - (k-h)(2k-1)|T| - (2k-1)(2m-1)\\ &\geq k((k-1)(n+2)+2m+1-h(2k-1)) - (k-h)(k(n-2)-2m+1+h(2k-1)) - (2k-1)(2m-1)\\ &= (h-1)(kn-(2k-1)(2k-h)-2m)+2k-1\end{aligned}$$

that is,

$$0 \geq (h-1)(kn-(2k-1)(2k-h)-2m)+2k-1. \tag{8}$$

Subcase 1.1.1. $h = 2$.
Using (8) and $n \geq 4k-6+\frac{2m}{k-1}$, we get

$$\begin{aligned}0 &\geq (h-1)(kn-(2k-1)(2k-h)-2m)+2k-1\\ &= kn-(2k-1)(2k-2)-2m+2k-1\\ &\geq k\left(4k-6+\frac{2m}{k-1}\right)-(2k-1)(2k-2)-2m+2k-1\\ &\geq 2k-3 \geq 1,\end{aligned}$$

which is a contradiction.
Subcase 1.1.2. $3 \leq h \leq k-1$.
In terms of (8) and $n \geq 4k-6+\frac{2m}{k-1}$, we have

$$\begin{aligned}0 &\geq (h-1)(kn-(2k-1)(2k-h)-2m)+2k-1\\ &\geq (h-1)(kn-(2k-1)(2k-3)-2m)+2k-1\\ &\geq (h-1)\left(4k^2-6k+\frac{2km}{k-1}-(2k-1)(2k-3)-2m\right)+2k-1\\ &\geq (h-1)(2k-3)+2k-1\\ &\geq 2(2k-3)+2k-1 = 6k-7 > 0.\end{aligned}$$

Which is a contradiction.
Subcase 1.2. $|T| \geq \frac{k(n-2)-2m+2}{2k-1}+h$.
Set $Y = T - N_{G-S}(x_1)$. Note that $|N_{G-S}(x_1)| = d_{G-S}(x_1)$. Thus, we have

$$\begin{aligned}|Y| &\geq |T| - d_{G-S}(x_1)\\ &= |T| - h \geq \frac{k(n-2)-2m+2}{2k-1} > 0\end{aligned}$$

and

$$N_G(Y) \neq V(G).$$

Combining these with the definition of $\mathrm{bind}(G)$, we obtain

$$\mathrm{bind}(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{n-1}{|T|-h} \leq \frac{(2k-1)(n-1)}{k(n-2)-2m+2}.$$

Which contradicts $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)-2m+2}$.
Case 2. $h = 1$.
Using (4) and $|S|+|T| \leq n$, we have

$$\begin{aligned}\delta_G(S,T) &= k|S| + \sum_{x\in T} d_{G-S}(x) - k|T|\\ &\geq k|S| + |T| - k|T| = k|S| - (k-1)|T|\\ &\geq k|S| - (k-1)(n-|S|) = (2k-1)|S| - (k-1)n\\ &> (k-1)(n+2)+2m-(2k-1)-(k-1)n\\ &= 2m-1,\end{aligned}$$

which contradicts (2).
Case 3. $h = 0$.
Put $\lambda = |\{x : x \in T, d_{G-S}(x) = 0\}|$ and $X = V(G) \setminus S$. Clearly, $\lambda \geq 1$ and $N_G(X) \neq V(G)$ since $h = 0$, and $|X| = |V(G) \setminus S| \geq |T| \geq 1$. Thus, by the definition of $\mathrm{bind}(G)$ we have

$$\frac{|N_G(X)|}{|X|} \geq \mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)-2m+2},$$

that is,

$$|N_G(X)| > \frac{(2k-1)(n-1)}{k(n-2)-2m+2}|X|. \tag{9}$$

On the other hand, it is easy to see that

$$|N_G(X)| \le n-\lambda. \tag{10}$$

From (9), (10) and $|X| = n - |S|$, we obtain

$$n-\lambda > \frac{(2k-1)(n-1)}{k(n-2)-2m+2}|X| = \frac{(2k-1)(n-1)}{k(n-2)-2m+2}(n-|S|),$$

which implies

$$|S| > n - \frac{(n-\lambda)(k(n-2)-2m+2)}{(2k-1)(n-1)}. \tag{11}$$

According to (2) and $|S| + |T| \le n$, we have

$$\begin{aligned} 2m-1 \ge \delta_G(S,T) &= k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \\ &\ge k|S| + |T| - \lambda - k|T| \\ &= k|S| - (k-1)|T| - \lambda \\ &\ge k|S| - (k-1)(n-|S|) - \lambda \\ &= (2k-1)|S| - (k-1)n - \lambda, \end{aligned}$$

which implies

$$|S| \le \frac{(k-1)n + 2m - 1 + \lambda}{2k-1} = n - \frac{kn - 2m + 1 - \lambda}{2k-1}. \tag{12}$$

From (11) and (12), we obtain

$$(n-\lambda)(k(n-2)-2m+2) > (n-1)(kn-2m+1-\lambda). \tag{13}$$

If the LHS and RHS of (13) are denoted by A and B respectively, then (13) says that $A - B > 0$. But, after some rearranging, we find that

$$A - B = -(k-1)n - \lambda((k-1)n - 2k - 2m + 3) - 2m + 1. \tag{14}$$

Since $n \ge 4k - 6 + \frac{2m}{k-1}$ and $\lambda \ge 1$, it is clear that the expression in (14) is negative, and this contradicts (13).

From the contradictions we deduce that G is a fractional (k, m)-deleted graph. This completes the proof of Theorem 6. $\square$

Remark. Let us show that the condition $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)-2m+2}$ in Theorem 6 cannot be replaced by $\mathrm{bind}(G) \ge \frac{(2k-1)(n-1)}{k(n-2)-2m+2}$. Let $k \ge 2$ and $0 \le m \le 2k-1$ be two integers such that m is odd and $\frac{(k+1)m}{k}$ is an integer, and let $l = \frac{2k+m-1}{2}$ and $m = 2k-3+\frac{(k+1)m}{k}$. We write $n = m+2l = 4k-4+\frac{(2k+1)m}{k}$. Obviously, n is a positive integer. Let $G = K_m \vee lK_2$ and $X = V(lK_2)$. Then for any $x \in X$, $|N_G(X \setminus x)| = n-1$. By the definition of $\mathrm{bind}(G)$, $\mathrm{bind}(G) = \frac{|N_G(X\setminus x)|}{|X\setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{2k+m-2} = \frac{(2k-1)(n-1)}{k(n-2)-2m+2}$. Let $S = V(K_m) \subseteq V(G)$, $T = V(lK_2) \subseteq V(G)$ and H is any subgraph of $G[T]$ with m edges. Then $|S| = m$, $|T| = 2l$ and $\sum_{x\in T} d_H(x) - e_H(S,T) = 2m$. Thus, we obtain

$$\begin{aligned} \delta_G(S,T) &= k|S| - k|T| + d_{G-S}(T) \\ &= k|S| - k|T| + |T| = k|S| - (k-1)|T| \\ &= km - 2(k-1)l \\ &= k\left(2k-3+\frac{(k+1)m}{k}\right) - (k-1)(2k+m-1) \\ &= 2m-1 < 2m = \sum_{x\in T} d_H(x) - e_H(S,T). \end{aligned}$$

In terms of Lemma 2.2, G is not a fractional (k, m)-deleted graph. In the above sense, the result in Theorem 6 is best possible.

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